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COMMENT

Comment on 'Exact analytical solution for the generalized Lyapunov exponent of the two-dimensional Anderson localization'

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Abstract

In a recent publication, Kuzovkov *et al* (2002 *J. Phys.: Condens. Matter.* **14** 13777) announced an analytical solution of the two-dimensional Anderson localization problem via the calculation of a generalized Lyapunov exponent using signal theory. Surprisingly, for certain energies and small disorder strength they observed delocalized states. We study the transmission properties of the same model using well-known transfer matrix methods. Our results disagree with the findings obtained using signal theory. We point to the possible origin of this discrepancy and comment on the general strategy of using a generalized Lyapunov exponent for studying Anderson localization.

It is generally believed that, in the absence of both spin–orbit scattering and magnetic fields, there is no metal–insulator transition for non-interacting electrons in disordered two-dimensional (2D) systems. This belief is based on the scaling hypothesis [1] and supported by detailed finite-size scaling analysis of numerical data [2]. Nevertheless, an analytical proof that there are no extended states in a disordered 2D system described by the Anderson model is still missing.

Recently, Kuzovkov *et al* [3] studied disordered 2D systems using signal theory and found parameter regimes where the system under consideration transmits the incident signal. They interpret this result as an indication for a metallic phase. This strongly contradicts standard wisdom and, therefore, in this comment, we reconsider the transmission properties of a 2D Anderson model using well-known transfer matrix methods. Similar calculations were presented before by Pendry [4].

Consider a 2D Anderson model on the $M \times L$ lattice (M is the width and L is the length of the system). The discrete Schrödinger equation

$$\psi_{n+1,m} = (E - \varepsilon_{nm})\psi_{nm} - \psi_{n-1,m} - \psi_{n,m+1} - \psi_{n,m-1} \tag{1}$$

may be rewritten using the transfer matrix $T_n^{(1)}$ as

$$\begin{pmatrix} \Psi_{n+1} \\ \Psi_n \end{pmatrix} = T_n^{(1)} \begin{pmatrix} \Psi_n \\ \Psi_{n-1} \end{pmatrix}, \qquad T_n^{(1)} = \begin{pmatrix} E - H_0 - \epsilon_n & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Here, $H_n = H_0 + \epsilon_n$ is the $(M \times M)$ Hamiltonian of the nth slice which contains random (uncorrelated) energies ε_{nm} and Ψ_n is the vector $(\psi_{n1}, \psi_{n2}, \ldots, \psi_{nM})$. For simplicity we assume for the disorder potentials that $\langle \varepsilon_{nm} \rangle = 0$ and $\langle \varepsilon_{nm} \varepsilon_{n'm'} \rangle = \sigma^2 \delta_{nn'} \delta_{mm'}$. The angle brackets denote averages over disorder.

In order to study the properties of the quantities $\langle \psi_{nm} \psi_{nm}^* \rangle$ (signals) as considered by Kuzovkov *et al* [3] we construct the tensor product of equation (2):

$$\begin{pmatrix} \Psi_{n+1} \\ \Psi_n \end{pmatrix} \otimes \begin{pmatrix} \Psi_{n+1}^* \\ \Psi_n^* \end{pmatrix} = T_n^{(2)} \begin{pmatrix} \Psi_n \\ \Psi_{n-1} \end{pmatrix} \otimes \begin{pmatrix} \Psi_n^* \\ \Psi_{n-1}^* \end{pmatrix}. \tag{3}$$

The size of the matrix $T_n^{(2)} = T_n^{(1)} \otimes T_n^{(1)}$ is $4M^2 \times 4M^2$. We may now average equation (3) over the disorder. Since the matrix $T_n^{(2)}$ and the tensor product on which it operates are statistically independent, we obtain a non-random matrix $T^{(2)} = \langle T_n^{(2)} \rangle$:

$$T^{(2)} = \begin{pmatrix} \sigma^2 1 \otimes 1 + D_0 \otimes D_0 & -D_0 \otimes 1 & -1 \otimes D_0 & 1 \\ D_0 \otimes 1 & 0 & -1 & 0 \\ 1 \otimes D_0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

with $D_0 = E - H_0$. The matrix (4) can be transformed by the matrix $Q \otimes Q$, where Q diagonalizes H_0. Using Dirichlet boundary conditions in the transversal direction, one obtains

$$Q D_0 Q^{-1} = \kappa, \qquad \kappa_i = E - 2\cos k_i, \qquad k_i = \frac{\pi}{M+1} i, \qquad i = 1, \ldots, M. \tag{5}$$

As a result, the $4M^2$ eigenvalues λ of $T^{(2)}$ may be calculated from

$$\prod_{i,j}^{M} \det \begin{pmatrix} \kappa_i \kappa_j + \sigma^2 - \lambda & -\kappa_i & -\kappa_j & 1 \\ \kappa_i & -\lambda & -1 & 0 \\ \kappa_j & -1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} = 0. \tag{6}$$

Therefore, the eigenvalues fulfil the equations

$$\lambda^4 - \lambda^3(\kappa_i \kappa_j + \sigma^2) + \lambda^2(\kappa_i^2 + \kappa_j^2 - 2) - \lambda(\kappa_i \kappa_j - \sigma^2) + 1 = 0. \tag{7}$$

Using $\lambda = \exp(\mathrm{i}q)$, equation (7) is

$$2\cos 2q - 2\kappa_i \kappa_j \cos q + (\kappa_i^2 + \kappa_j^2 - 2) = 2\sigma^2 \mathrm{i} \sin q. \tag{8}$$

For the ordered case ($\sigma = 0$) the $4M^2$ solutions are obtained as $q_{ij} = \pm a_i \pm a_j$ with $2\cos a_i = E - 2\cos k_i$ and the eigenvalues of the unperturbed matrix are given by

$$\lambda(\sigma^2 = 0) = \exp\bigl[\mathrm{i}(\pm a_i \pm a_j)\bigr]. \tag{9}$$

If $\sigma \neq 0$, then equation (8) shows that q is real (i.e. $|\lambda| = 1$) only if q is zero or a multiple of π. Therefore, the only eigenvalues of the transfer matrix $T^{(2)}$ which lie on the unit circle are $\lambda = \pm 1$. Note that these eigenvalues are independent of the disorder. Therefore, it is evident that they correspond to some internal symmetry of the Anderson model (see also [4]). In fact, from equation (7) one finds M eigenvalues $\lambda = 1$ (one for each $i = j$). They just correspond to current conservation in each of the M channels [4]. We will call these eigenvalues 'trivial eigenvalues'. Eigenvalues $\lambda = -1$ are obtained for $\kappa_i = -\kappa_j$ corresponding to the special energy $E = \cos k_i + \cos k_j$, which can be discarded as irrelevant.

Our results so far do not disagree with those of [3]. However, Kuzovkov *et al* claim that in the limit $M \to \infty$ eigenvalues $|\lambda| = 1$ will appear, which do not correspond to the

trivial eigenvalues discussed above. Mathematically, the limit $M \to \infty$ of the discrete model discussed here bears various conceptual and technical difficulties. Therefore we studied this limit numerically and find no new solutions with $|\lambda| = 1$ apart from the trivial ones. It therefore appears that the metallic solutions found by Kuzovkov *et al* are an artefact of a questionable mathematical limit procedure.

In conclusion, our analysis shows that, apart from the 'trivial' eigenvalues $\lambda = \pm 1$, which correspond to symmetries of the 2D Anderson Hamiltonian, all other eigenvalues have an absolute value different from unity. This holds for any M. Therefore, all input signals are damped away if the system is sufficiently long. This result contradicts the findings in [3]. We believe that the reasons for this contradiction are as follows: Kuzovkov *et al* reduced the size of the transfer matrix from $4M^2 \times 4M^2$ to only $\sim M$. This drastic reduction was achieved by introducing a translational symmetry in the transversal direction. A precise description and motivation of this additional averaging procedure is absent in their paper. We do not believe that such a symmetry really holds in general. We therefore have to conclude that, by this reduction of the size of the transfer matrix, they transformed the model defined in equation (1) into another model. But even then Kuzovkov *et al* do not find eigenvalues $|\lambda| = 1$ different from the trivial eigenvalues. Only by using a questionable limit procedure $M \to \infty$ are such eigenvalues obtained.

Finally, we would like to comment on the general strategy to tackle the Anderson localization problem using methods similar to the one discussed here and in [3]. We do not think that the analysis of the second moments of the wavefunction $|\psi|^2$ (or of any higher moments $|\psi|^{2m}$) as was done here as well as in [3] is able to detect a metallic phase. The physical reason is the following: in order to detect a metallic phase one needs to average the logarithm of the wavefunction itself [5]. The procedures discussed here, while mathematically correct, always lead to quantities which are damped away for large systems, i.e. yield eigenvalues $|\lambda| \neq 1$, apart from the trivial ones. This can be exemplified most drastically with an analysis of a 3D system: here, one obtains equation (6) with $\kappa_i = E - 2\cos k_\alpha - 2\cos k_\beta$, $i = M(\alpha - 1) + \beta$, $\alpha, \beta = 1, \ldots, M$. Consequently, the absolute values of all non-trivial eigenvalues differ from unity, independently of the strength of the disorder. Since weakly disordered 3D systems are metallic, we conclude that, in contrast to $\langle \ln(|\psi|) \rangle$, an analysis of $\langle |\psi|^2 \rangle$ does not help to observe a metallic regime even if it exists.

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